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Quantum Inozemtsev model, quasi-exact solvability and $\mathcal{N}$-fold supersymmetry

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Abstract

Inozemtsev models are classically integrable multi-particle dynamical systems related to Calogero–Moser (CM) models. Because of the additional q^6 (rational models) or $\sin^2 2q$ (trigonometric models) potentials, their quantum versions are not exactly solvable, in contrast to CM models. We show that quantum Inozemtsev models can be deformed to be a widest class of partly solvable (or quasi-exactly solvable) multi-particle dynamical systems. They posses $\mathcal{N}$-fold supersymmetry, which is equivalent to quasi-exact solvability. A new method for identifying and solving quasi-exactly solvable systems, the method of pre-superpotential, is presented.

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1. Introduction

In this paper we address the problem of the relationship amongst classical integrability, quantum integrability and quantum partial integrability (or quasi-exact solvability) within multi-particle dynamical systems by taking a wide class of explicit examples, the Inozemtsev models. We demonstrate that the Inozemtsev models (with degenerate potentials, that is, non-elliptic potentials) can be made quantum partly solvable (or QES (quasi-exactly solvable)). For this purpose we present a simple new formulation of QES systems of single as well as multiple degrees of freedom. We also show that the notions of higher derivative (or nonlinear or $\mathcal{N}$-fold) supersymmetry and quasi-exact solvability are equivalent. In other words, Inozemtsev models provide plentiful examples of multi-particle $\mathcal{N}$-fold supersymmetry.

Inozemtsev models [1–5] are a generalization of Calogero–Moser (CM) models [6–13] associated with the root system of BC type and A type. They belong to the category of 'twisted' CM models [12, 14]. Like all the CM models they are classically integrable for all four types of potentials (elliptic, hyperbolic, trigonometric and rational) in the sense that their equations of motion can be expressed in Lax pair forms.

We start by asking a general and naturally vague question: to what extent does classical integrability imply quantum integrability in multi-particle dynamical systems? As is well known, the converse, that quantum integrability always implies classical integrability, is true in multi-particle dynamical systems, since the quantum system is the $\hbar$ deformation of the classical one. We do not know a way to answer this question abstractly by starting from the pure notion of classical integrability, although some attempts have been made to construct quantum conserved quantities as a deformation of classical ones in the framework of perturbed conformal field theory (see, e.g., [15]).

However, the experience of CM models, the widest class of integrable multi-particle systems ever known, tells us that classical integrability is very close to quantum integrability. The quantum integrability is proved universally, that is, for all the root systems including the non-crystallographic ones, for CM models with degenerate potentials [16, 17], namely those with trigonometric, hyperbolic and rational potentials.

Thus we are naturally led to the question of quantum integrability of Inozemtsev models with degenerate potentials. Rational Inozemtsev models have an additional potential of sixth degree polynomial in q (coordinates), see (3.10), on top of the CM potentials. Trigonometric Inozemtsev models have an additional $\sin^2 2q$ potential, see (4.11), on top of the CM potentials. These additional potentials destroy the mechanism for providing quantum conserved quantities, the so-called 'sum-to-zero' condition of the second member of the Lax pair [16, 18].

The very interactions (q^6, $\sin^2 2q$, etc) that constitute obstructions for quantum integrability of Inozemtsev models are known to play essential roles in the QES [19] one-particle quantum mechanics. This leads to a conjecture that *at least a certain class of Inozemtsev models can be made QES*. We demonstrate that *supersymmetrizable* (see (2.5) and (2.6)) Inozemtsev models can be deformed to QES systems which are characterized by an integer deformation parameter $\mathcal{M}$. It is also shown that the concepts of quasi-exact solvability and *higher-derivative* [20] or *nonlinear* [21] or *$\mathcal{N}$-fold* [22] supersymmetry (with a typical relationship $\mathcal{N} = \mathcal{M} + 1$) are equivalent.

This paper is organized as follows. In section 2 we present the basic tool for investigating QES systems which we call the *method of pre-superpotential*. We briefly summarize the classical Inozemtsev models in comparison to CM models. In section 3 we demonstrate the quasi-exact solvability of a single-particle rational BC type Inozemtsev model based on a new method of employing a pre-superpotential W. This provides the most general single-particle QES system with q^6 potential known to date. The equivalence of quasi-exact solvability and $\mathcal{N}$-fold supersymmetry is generally established. Other related notions, the 'Bethe ansatz' type equations [23], ODE spectral equivalence [24, 25] and Bender–Dunne polynomials [26], are simply explained from the new point of view. Section 4 deals with the quasi-exact solvability of a single-particle trigonometric BC type Inozemtsev model. In section 5 we discuss the rational A type Inozemtsev model with q^4 potential, which provides an example of a spontaneously broken $\mathcal{N}$-fold supersymmetry. In sections 6–8, various Inozemtsev models (rational BC type, trigonometric BC type and trigonometric A type) are shown to be QES and the generators of $\mathcal{N}$-fold supersymmetries are identified. The QES of quantum Inozemtsev models is the consequence of exact solvability of quantum CM models and QES of the added single-particle type interactions. The final section is devoted to comments and discussion. Appendix A gives the classical Lax pairs for the BC type and A type Inozemtsev models in the same notation as used in the main text. Appendix B presents details of the lower triangularity of trigonometric CM interactions which are necessary for establishing quasi-exact solvability of trigonometric Inozemtsev models.

2. Basic tool and classical Inozemtsev models

2.1. Basic tool

One basic tool for showing the existence of some exact eigenfunctions (quasi-exact solvability) is the following simple fact. Let $W = W(q)$ be a real smooth function of the coordinate(s); then trivial differentiation formulae ($p = -\mathrm{i}\partial/\partial q$):

$$p^2 \mathrm{e}^W = -\left[\left(\frac{\partial W}{\partial q}\right)^2 + \frac{\partial^2 W}{\partial q^2}\right]\mathrm{e}^W \qquad \sum_{j=1}^r p_j^2\, \mathrm{e}^W = -\sum_{j=1}^r \left[\left(\frac{\partial W}{\partial q_j}\right)^2 + \frac{\partial^2 W}{\partial q_j^2}\right]\mathrm{e}^W$$

imply that e^W is an eigenfunction of the Hamiltonian H with eigenvalue 0:

$$H\,\mathrm{e}^W = 0$$
$$H = \frac{1}{2}p^2 + \frac{1}{2}\left[\left(\frac{\partial W}{\partial q}\right)^2 + \frac{\partial^2 W}{\partial q^2}\right] \qquad H = \frac{1}{2}\sum_{j=1}^r p_j^2 + \frac{1}{2}\sum_{j=1}^r\left[\left(\frac{\partial W}{\partial q_j}\right)^2 + \frac{\partial^2 W}{\partial q_j^2}\right] \tag{2.1}$$

so long as it is square integrable:

$$\int \mathrm{e}^{2W}\,\mathrm{d}^r q < \infty. \tag{2.2}$$

This is the simplest example of quasi-exact solvability. Looked at differently, one might say this is a property of 'factorized' Hamiltonians:

$$H = \frac{1}{2}\left(p - \mathrm{i}\frac{\partial W}{\partial q}\right)\left(p + \mathrm{i}\frac{\partial W}{\partial q}\right) \qquad H = \frac{1}{2}\sum_{j=1}^r\left(p_j - \mathrm{i}\frac{\partial W}{\partial q_j}\right)\left(p_j + \mathrm{i}\frac{\partial W}{\partial q_j}\right) \tag{2.3}$$

together with a differential operator(s) that annihilates the state

$$\left(p + \mathrm{i}\frac{\partial W}{\partial q}\right)\mathrm{e}^W = 0 \qquad \left(p_j + \mathrm{i}\frac{\partial W}{\partial q_j}\right)\mathrm{e}^W = 0 \qquad j = 1, \ldots, r. \tag{2.4}$$

This fact can also be considered as the very base of supersymmetric quantum mechanics.

This gives the ground-state wavefunction of the quantum CM models. In other words, all the (quantum integrable) CM models can be described by pre-superpotentials W [10, 18]. To sum up, if a Hamiltonian can be expressed in terms of W as (2.1) or (2.3) up to a constant, the existence of one eigenfunction is guaranteed save the square normalizability. Throughout this paper we call function W a *pre-superpotential*.

2.2. Classical models

Here we present classical Inozemtsev models together with CM models for comparison. The dynamical variables are canonical coordinates $\{q_j\}$ and their canonical conjugate momenta $\{p_j\}$. We denote them by r-dimensional vectors q and p with standard inner product:

$$q = (q_1, \ldots, q_r) \in \boldsymbol{R}^r \qquad p = (p_1, \ldots, p_r) \in \boldsymbol{R}^r \qquad p^2 = p \cdot p = p_1^2 + \cdots + p_r^2.$$

As is well known for a root system Δ (rank r) and four types of potentials (elliptic, trigonometric, hyperbolic and rational), CM and Inozemtsev models are classically completely integrable. In this paper we discuss only those models based on classical root systems, that is, A type and BC (and D) type, and *degenerate* potentials, that is, trigonometric, hyperbolic and rational potentials. Since algebraic structures are almost the same for the trigonometric and hyperbolic potential models, we discuss the trigonometric case as a representative. Among various types of Inozemtsev models [1–5] we focus our attention on the *supersymmetrizable* models, namely on those models whose Hamiltonians can be collectively expressed in terms of a pre-superpotential $W = W(q)$ [18] as

$$H = \frac{1}{2}p^2 + \frac{1}{2}\sum_{j=1}^{r}\left(\frac{\partial W}{\partial q_j}\right)^2 \tag{2.5}$$

or 'factorizable' at the classical level:

$$H = \frac{1}{2}\sum_{j=1}^{r}\left(p_j - \mathrm{i}\frac{\partial W}{\partial q_j}\right)\left(p_j + \mathrm{i}\frac{\partial W}{\partial q_j}\right). \tag{2.6}$$

Each specific model in this class is given by a choice of W, which are listed below.

2.3. *BC type Calogero–Moser models*

The rational model pre-superpotential W:

$$W_{\mathrm{CM}} = g_M \sum_{j<k}^{r}\left\{\log|q_j - q_k| + \log|q_j + q_k|\right\} + g_S \sum_{j=1}^{r}\log|q_j| \tag{2.7}$$

contains two real coupling constants g_M for the middle roots (length$^2 = 2$) and g_S for the short roots (length$^2 = 1$). The trigonometric model pre-superpotential W:

$$W_{\mathrm{CM}} = g_M \sum_{j<k}^{r}\left\{\log|\sin(q_j - q_k)| + \log|\sin(q_j + q_k)|\right\}$$
$$+ g_S \sum_{j=1}^{r}\log|\sin q_j| + g_L \sum_{j=1}^{r}\log|\sin 2q_j| \tag{2.8}$$

has one more coupling constant than the rational case, g_L for the long roots (length$^2 = 4$). For the rational potential, the long roots and short roots are essentially the same. When $g_L = 0$ and $g_S = 0$ the models belong to the D_r root system. If $g_L = 0$ and $g_S \neq 0$ ($g_L \neq 0$ and $g_S = 0$) the models belong to the B_r (C_r) root system. Throughout this paper we put the scale factor in the trigonometric functions to unity for simplicity.

2.4. *A type Calogero–Moser models*

For A type models, it is customary to express the roots by embedding in a space with one higher dimension. We will discuss A_{r-1} models with r degrees of freedom. Since all the roots have the same length, A type models have only one real coupling constant g. The rational model pre-superpotential W is given by

$$W_{\mathrm{CM}} = g \sum_{j<k}^{r}\log|q_j - q_k| \tag{2.9}$$

whereas the trigonometric pre-superpotential W is

$$W_{\mathrm{CM}} = g \sum_{j<k}^{r}\log|\sin(q_j - q_k)|. \tag{2.10}$$

2.5. BC type Inozemtsev models

The rational supersymmetrizable BC type Inozemtsev model [4,5] has two more real coupling constants, a and b, than the corresponding CM model:

$$W = -\sum_{j=1}^{r}\left(\frac{a}{4}q_j^4 + \frac{b}{2}q_j^2\right) + W_{\mathrm{CM}}\ (2.7) \tag{2.11}$$

which leads to degree six polynomial potentials. The trigonometric supersymmetrizable BC type Inozemtsev model [4, 5] also has two more real coupling constants, a and b, than the corresponding CM model:

$$W = \sum_{j=1}^{r}\left(-\frac{a}{2}\cos 2q_j + \frac{b}{2}\log|\cot q_j|\right) + W_{\mathrm{CM}}\ (2.8). \tag{2.12}$$

2.6. A type Inozemtsev models

The rational supersymmetrizable A type Inozemtsev model [1–3] has two more real coupling constants, a and b, than the corresponding CM model:

$$W = \sum_{j=1}^{r}\left(\frac{a}{3}q_j^3 + \frac{b}{2}q_j^2\right) + W_{\mathrm{CM}}\ (2.9) \tag{2.13}$$

which leads to degree four polynomial potentials. The trigonometric A type Inozemtsev model [1–3] has only one more real coupling constant, a, than the corresponding CM model:

$$W = \sum_{j=1}^{r}\left(-\frac{a}{2}\cos 2q_j\right) + W_{\mathrm{CM}}\ (2.10). \tag{2.14}$$

It should be emphasized that these additional interactions of the supersymmetrizable Inozemtsev models are all 'single-particle' type. In the following three sections 3–5 we will investigate the characteristic single-particle dynamics of supersymmetrizable Inozemtsev models.

3. Rational BC type Inozemtsev model with one degree of freedom

Let us start with a pre-superpotential $W = W(q)$ which is decomposed into two parts:

$$W = W_0 + W_1 \tag{3.1}$$

$$W_0 = -\left(\frac{a}{4}q^4 + \frac{b}{2}q^2\right) + g_S \log|q| \qquad a > 0 \quad g_S > 0 \tag{3.2}$$

$$W_1 = \sum_{k=1}^{\mathcal{M}} \log|q^2 - \xi_k| \tag{3.3}$$

in which $\mathcal{M}$ is an arbitrary non-negative integer and $\{\xi_k\}$ are distinct but as yet undetermined parameters. The first part W_0 corresponds to the 'single-particle interactions' of the BC type rational Inozemtsev model, which is an even function of q. The added part gives rise to an arbitrary polynomial in q^2 of degree $\mathcal{M}$ in $\mathrm{e}^W = \mathrm{e}^{W_0}\prod_{k=1}^{\mathcal{M}}(q^2 - \xi_k)$. Since

$$\left(\frac{\partial W}{\partial q}\right)^2 + \frac{\partial^2 W}{\partial q^2} = \left(\frac{\partial W_0}{\partial q}\right)^2 + \frac{\partial^2 W_0}{\partial q^2} + 2\frac{\partial W_0}{\partial q}\frac{\partial W_1}{\partial q} + \left(\frac{\partial W_1}{\partial q}\right)^2 + \frac{\partial^2 W_1}{\partial q^2}$$

we will evaluate the terms containing W_1:

$$2\frac{\partial W_0}{\partial q}\frac{\partial W_1}{\partial q}+\left(\frac{\partial W_1}{\partial q}\right)^2+\frac{\partial^2 W_1}{\partial q^2}=2\left\{-(aq^3+bq)+\frac{g_S}{q}\right\}\sum_{k=1}^{\mathcal{M}}\frac{2q}{q^2-\xi_k}$$
$$+\sum_{k=1}^{\mathcal{M}}\frac{2}{q^2-\xi_k}+8q^2\sum_{k<l}\frac{1}{q^2-\xi_k}\frac{1}{q^2-\xi_l}. \qquad (3.4)$$

This is a meromorphic function in q^2 with at most simple poles. We demand that the residues of the simple poles, $q^2=\xi_k$, $k=1,\ldots,\mathcal{M}$ should all vanish [23], which results in a set of rational ('Bethe ansatz' type) equations for $\{\xi_k\}$:

$$2\left\{-(a\xi_k^2+b\xi_k)+g_S\right\}+1+4\xi_k\sum_{l\neq k}\frac{1}{\xi_k-\xi_l}=0 \qquad k=1,\ldots,\mathcal{M}. \qquad (3.5)$$

Then expression (3.4) becomes a linear polynomial in q^2, which is easy to evaluate:

$$(3.4)=-4a\mathcal{M}q^2-4bM-4a\sum_{k=1}^{\mathcal{M}}\xi_k.$$

Thus we arrive at

$$\left(\frac{\partial W}{\partial q}\right)^2+\frac{\partial^2 W}{\partial q^2}=\left(\frac{\partial W_0}{\partial q}\right)^2+\frac{\partial^2 W_0}{\partial q^2}-4a\mathcal{M}q^2-2E_1 \qquad (3.6)$$

$$E_1=2bM+2a\sum_{k=1}^{\mathcal{M}}\xi_k. \qquad (3.7)$$

It should be emphasized that, except for the constant term E_1, the expression (3.6) is independent of the parameters $\{\xi_k\}$ introduced in W_1.

This means that, for each set of solutions $\{\xi_k\}$ (with real $\sum\xi_k$ and up to the ordering), we have an eigenstate

$$\mathrm{e}^W=\mathrm{e}^{W_0}\prod_{k=1}^{\mathcal{M}}(q^2-\xi_k)=q^{g_S}\,\mathrm{e}^{-(aq^4/4+bq^2/2)}\prod_{k=1}^{\mathcal{M}}(q^2-\xi_k) \qquad (3.8)$$

with eigenvalue E_1, (3.7) of the Hamiltonian

$$H=\frac{1}{2}p^2+\frac{1}{2}\left[\left(\frac{\partial W_0}{\partial q}\right)^2+\frac{\partial^2 W_0}{\partial q^2}\right]-2a\mathcal{M}q^2 \qquad (3.9)$$
$$=\frac{1}{2}p^2+\frac{1}{2}q^2(aq^2+b)^2+\frac{g_S(g_S-1)}{2q^2}-a\left(\frac{3}{2}+2\mathcal{M}+g_S\right)q^2-\frac{b}{2}(1+2g_S). \qquad (3.10)$$

It has q^6, q^4, q^2 and $1/q^2$ potentials and a part of the coefficients of the quadratic potential is *quantized*. Because of the singular centrifugal term $1/q^2$, we restrict the function space to the half line, $(0,+\infty)$. The restriction on the coupling constants, $a>0$, is for securing the square integrability of e^W at $q=+\infty$ and $g_S>0$ for finiteness at $q=0$.

The above result implies that, by adding a single term $-2a\mathcal{M}q^2$ to the Hamiltonian, the single-particle BC type rational Inozemtsev model can be made *QES*, which means a finite number of eigenstates together with their eigenvalues can be obtained exactly by algebraic means. The very term $-\frac{a}{4}q^4$ in W_0 that obstructs quantum integrability is instrumental for the introduction of the additional term $-2a\mathcal{M}q^2$. The eigenfunction (3.8) of the above *QES* system belongs to a 'polynomial space'

$$\mathcal{V}_{\mathcal{M}}=\mathrm{Span}\big[1,q^2,\ldots,q^{2k},\ldots,q^{2\mathcal{M}}\big]\mathrm{e}^{W_0}. \qquad (3.11)$$

In other words, the Hamiltonian (3.10) leaves this polynomial space invariant:

$$H\mathcal{V}_{\mathcal{M}} \subseteq \mathcal{V}_{\mathcal{M}}. \tag{3.12}$$

Therefore this 'polynomial space' can be called the *exactly solvable sector* of the system. It is elementary to see that the 'polynomial space' $\mathcal{V}_{\mathcal{M}}$ is annihilated by an $\mathcal{N} = (\mathcal{M}+1)$th-order differential operator $P_{\mathcal{N}}$:

$$P_{\mathcal{N}} = \prod_{k=0}^{\mathcal{N}-1}\left(D + \mathrm{i}\frac{k}{q}\right) = \left(D + \frac{\mathrm{i}(\mathcal{N}-1)}{q}\right)\cdots\left(D + \frac{\mathrm{i}}{q}\right)D \tag{3.13}$$

$$P_{\mathcal{N}}\mathcal{V}_{\mathcal{M}} = 0 \qquad D = p + \mathrm{i}\frac{\partial W_0}{\partial q}. \tag{3.14}$$

Since $P_{\mathcal{N}}$ is an $\mathcal{N} = (\mathcal{M}+1)$th-order differential operator, it is obvious that $\mathcal{V}_{\mathcal{M}}$ gives the entire solution space of a differential equation

$$P_{\mathcal{N}}y = 0.$$

This differential operator, together with its Hermitian conjugate, defines a *higher-derivative* [20] or *nonlinear* [21] or $\mathcal{N}$*-fold* [22] supersymmetry generated by

$$Q = P_{\mathcal{N}}\psi^{\dagger} \qquad Q^{\dagger} = P_{\mathcal{N}}^{\dagger}\psi \tag{3.15}$$

in which ψ and $\psi^{\dagger}$ are fermion annihilation and creation operators. The 'polynomial space' $\mathcal{V}_{\mathcal{M}}$ is characterized as the zero modes of Q and $Q^{\dagger}$:

$$Q\mathcal{V}_{\mathcal{M}} = Q^{\dagger}\mathcal{V}_{\mathcal{M}} = 0 \tag{3.16}$$

which is the generalization of the property of the ground state of the ordinary ($\mathcal{N} = 1$) supersymmetric quantum mechanics. The second equality $Q^{\dagger}\mathcal{V}_{\mathcal{M}} = 0$ is trivial since $\mathcal{V}_{\mathcal{M}}$ has zero fermion number.

The structure of the *exactly solvable sector* can be better understood by making a similarity transformation of H by e^{W_0} (see [16] for example):

$$\begin{aligned}\tilde{H} &= \mathrm{e}^{-W_0}H\,\mathrm{e}^{W_0} = \frac{1}{2}p^2 - \frac{\partial W_0}{\partial q}\frac{\partial}{\partial q} - 2a\mathcal{M}q^2 \\ &= \frac{1}{2}p^2 + \left(aq^3 + bq - \frac{g_S}{q}\right)\frac{\partial}{\partial q} - 2a\mathcal{M}q^2.\end{aligned} \tag{3.17}$$

Then the above 'polynomial space' (3.11) is mapped to a genuine polynomial space

$$\tilde{\mathcal{V}}_{\mathcal{M}} = \mathrm{Span}\left[1, q^2, \ldots, q^{2k}, \ldots, q^{2\mathcal{M}}\right] \tag{3.18}$$

whose invariance under $\tilde{H}$ (3.17)

$$\tilde{H}\tilde{\mathcal{V}}_{\mathcal{M}} \subseteq \tilde{\mathcal{V}}_{\mathcal{M}}$$

is rather elementary to verify. If one substitutes an expansion

$$\tilde{\Psi} = \sum_{k=0}^{\mathcal{M}}\alpha_k q^{2k} \qquad \alpha_0 = 1$$

into the eigenvalue equation

$$\tilde{H}\tilde{\Psi} = E\tilde{\Psi}$$

one obtains a three-term recursion relation for $\{\alpha_k\}$

$$(k+1)(2k+1+g_S)\alpha_{k+1} = (2kb - E)\alpha_k + 2a(k-\mathcal{M}-1)\alpha_{k-1}. \tag{3.19}$$

This determines α_k as a polynomial in E of degree k which is a Bender–Dunne polynomial [26] in the naivest sense. The condition $\alpha_{\mathcal{M}+1} = 0$ gives the characteristic equation of $\tilde{H}$:

$$\alpha_{\mathcal{M}+1} = 0 \Leftrightarrow (2\mathcal{M}b - E)\alpha_{\mathcal{M}} - 2a\alpha_{\mathcal{M}-1} = 0 \Leftrightarrow \det(\tilde{H} - E) = 0. \tag{3.20}$$

In the rest of this section let us discuss the relationship between quasi-exact solvability and $\mathcal{N}$-fold supersymmetry in the general context. This applies to the other cases discussed in the following sections as well. The *exactly solvable sector* of a QES theory is characterized by its 'polynomial space':

$$\mathcal{V}_{\mathcal{M}} = \mathrm{Span}\big[1, h, \ldots, h^k, \ldots, h^{\mathcal{M}}\big]\mathrm{e}^{W_{\mathrm{gen}}} \tag{3.21}$$

which is invariant under Hamiltonian

$$H_{\mathrm{gen}}\mathcal{V}_{\mathcal{M}} \subseteq \mathcal{V}_{\mathcal{M}}. \tag{3.22}$$

In these formulae the subscript 'gen' in H and W stands for 'generic' and the function $h = h(q)$ need not be a polynomial in q. (For example, in the trigonometric BC type Inozemtsev model (see section 4) $h(q) = \sin^2 q$.) It is straightforward to verify that the 'polynomial space' $\mathcal{V}_{\mathcal{M}}$ is annihilated by an $\mathcal{N} = (\mathcal{M}+1)$th-order differential operator $P_{\mathcal{N}}$:

$$P_{\mathcal{N}} = \prod_{k=0}^{\mathcal{N}-1}(D_{\mathrm{gen}} + \mathrm{i}kE(q)) \qquad D_{\mathrm{gen}} = p + \mathrm{i}\frac{\partial W_{\mathrm{gen}}}{\partial q} \quad E(q) \equiv \frac{h''(q)}{h'(q)} \tag{3.23}$$

$$P_{\mathcal{N}}\mathcal{V}_{\mathcal{M}} = 0. \tag{3.24}$$

As above, the 'polynomial space' $\mathcal{V}_{\mathcal{M}}$ gives the entire solution space of the differential equation $P_{\mathcal{N}}y = 0$. One could summarize the situation as *the exactly solvable sector of a QES dynamics is characterized as the states annihilated* (3.16) *by the generators Q and $Q^\dagger$* (3.15) *of an $\mathcal{N}$-fold supersymmetry*.

On the other hand, let us suppose that one has a pair of Hamiltonians H_{gen} and H^+_{gen} which are intertwined by $P_{\mathcal{N}}$ [22, 24, 25]:

$$P_{\mathcal{N}}H_{\mathrm{gen}} - H^+_{\mathrm{gen}}P_{\mathcal{N}} = 0. \tag{3.25}$$

Let $\mathcal{V}_{\mathcal{M}}$ be the space of solutions of the differential equation $P_{\mathcal{N}}y = 0$, which is finite dimensional. Then from (3.25) one obtains $P_{\mathcal{N}}H_{\mathrm{gen}}\mathcal{V}_{\mathcal{M}} = 0$ and thus deduces that the finite-dimensional space $\mathcal{V}_{\mathcal{M}}$ is invariant under H_{gen}: $H_{\mathrm{gen}}\mathcal{V}_{\mathcal{M}} \subseteq \mathcal{V}_{\mathcal{M}}$ (3.22). One could summarize this as *the quasi-exact solvability of H_{gen} is a consequence of the $\mathcal{N}$-fold supersymmetry and the intertwining relation* (3.25). The spectral equivalence of H_{gen} and H^+_{gen} holds outside $\mathcal{V}_{\mathcal{M}}$ as in the ordinary ($\mathcal{N} = 1$) supersymmetric quantum mechanics.

4. Trigonometric BC type Inozemtsev model with one degree of freedom

Here we consider a one-dimensional quantum mechanical system with the following pre-superpotential W in a finite interval $[0, \pi/2]$:

$$W = W_0 + W_1 \tag{4.1}$$

$$W_0 = -\frac{a}{2}\cos 2q + \frac{b}{2}\log|\cot q| + g_S\log|\sin q| \tag{4.2}$$

$$g_S > 0 \qquad g_S > \frac{b}{2} > 0 \tag{4.3}$$

$$W_1 = \sum_{k=1}^{\mathcal{M}}\log|\sin^2 q - \xi_k| \tag{4.4}$$

in which the W_0 part is obtained by retaining the single-particle part of the trigonometric BC type Inozemtsev model (2.12). It is an even function of q and it reduces to a well known 'double sine–Gordon' quantum mechanics [27] if only the first term $\frac{a}{2}\cos 2q$ is kept. Here we have not included the long root term $g_L \log|\sin 2q|$ in (2.8) since it can be expressed as a linear combination of $\log|\cot q|$ and $\log|\sin q|$ terms. As in the previous section, we evaluate the terms containing W_1 in $(\partial W/\partial q)^2 + \partial^2 W/\partial q^2$:

$$2\left(a\sin 2q - \frac{b}{\sin 2q} + g_S \cot q\right)\sum_{k=1}^{\mathcal{M}} \frac{2\sin q\cos q}{\sin^2 q - \xi_k}$$
$$+8\sin^2 q\cos^2 q \sum_{k<l} \frac{1}{\sin^2 q - \xi_k}\frac{1}{\sin^2 q - \xi_l} + 2\cos 2q \sum_{k=1}^{\mathcal{M}} \frac{1}{\sin^2 q - \xi_k} \tag{4.5}$$

which is a meromorphic function in $x = \sin^2 q$ with at most simple poles:

$$(4.5) = 2\left(4ax(1-x) - b + 2g_S(1-x)\right)\sum_{k=1}^{\mathcal{M}} \frac{1}{x-\xi_k}$$
$$+8x(1-x)\sum_{k<l}\frac{1}{x-\xi_k}\frac{1}{x-\xi_l} + 2(1-2x)\sum_{k=1}^{\mathcal{M}}\frac{1}{x-\xi_k}. \tag{4.6}$$

As in the previous case we demand that the residues at the simple poles $x = \xi_k, k = 1, \ldots, \mathcal{M}$ should all vanish. This requires that the $\{\xi_k\}$ should obey a set of rational equations:

$$(4a\xi_k + 2g_S)(1-\xi_k) - b + 1 - 2\xi_k + 4\xi_k(1-\xi_k)\sum_{l\neq k}\frac{1}{\xi_k - \xi_l} = 0 \qquad k = 1, \ldots, \mathcal{M}. \tag{4.7}$$

Then expression (4.6) becomes a linear function in $x = \sin^2 q$ which is easy to evaluate:

$$(4.5) = -8a\mathcal{M}\sin^2 q - 8a\sum_{k=1}^{\mathcal{M}}\xi_k - 4\mathcal{M}(g_S + \mathcal{M}).$$

Thus we arrive at

$$\left(\frac{\partial W}{\partial q}\right)^2 + \frac{\partial^2 W}{\partial q^2} = \left(\frac{\partial W_0}{\partial q}\right)^2 + \frac{\partial^2 W_0}{\partial q^2} - 8a\mathcal{M}\sin^2 q - 2E_1 \tag{4.8}$$

$$E_1 = 4a\sum_{k=1}^{\mathcal{M}}\xi_k + 2\mathcal{M}(g_S + \mathcal{M}). \tag{4.9}$$

Again, except for the constant term, expression (4.8) is independent of the parameters $\{\xi_k\}$. Thus, for each real solution of (4.7), we have an eigenfunction

$$e^W = e^{W_0}\prod_{k=1}^{\mathcal{M}}(\sin^2 q - \xi_k) = (\sin q)^{g_S}(\cot q)^{b/2}e^{-(a/2)\cos 2q}\prod_{k=1}^{\mathcal{M}}(\sin^2 q - \xi_k) \tag{4.10}$$

with eigenvalue E_1 (4.9), of the Hamiltonian

$$H = \frac{1}{2}p^2 + \frac{1}{2}\left[\left(\frac{\partial W_0}{\partial q}\right)^2 + \frac{\partial^2 W_0}{\partial q^2}\right] - 4a\mathcal{M}\sin^2 q. \tag{4.11}$$

The eigenfunction (4.10) is square integrable in $[0, \pi/2]$ due to the restriction on the parameters (4.3). In other words, the above Hamiltonian (4.11) is QES. The eigenfunction (4.10) belongs to a 'polynomial space':

$$\mathcal{V}_{\mathcal{M}} = \mathrm{Span}\left[1, \sin^2 q, \ldots, (\sin q)^{2k}, \ldots, (\sin q)^{2\mathcal{M}}\right]e^{W_0} \tag{4.12}$$

which is invariant under the action of the Hamiltonian $H\mathcal{V}_\mathcal{M} \subseteq \mathcal{V}_\mathcal{M}$. It is easy to see that $\mathcal{V}_\mathcal{M}$ is annihilated by an $\mathcal{N} = (\mathcal{M}+1)$th-order differential operator $P_\mathcal{N}$:

$$P_\mathcal{N} = \prod_{k=0}^{\mathcal{N}-1}(D + \mathrm{i}2k\cot 2q) \qquad D = p + \mathrm{i}\frac{\partial W_0}{\partial q} \tag{4.13}$$

$$P_\mathcal{N}\mathcal{V}_\mathcal{M} = 0. \tag{4.14}$$

Thus the general statements in the previous section concerning the quasi-exact solvability and $\mathcal{N}$-fold supersymmetry also hold in this case.

In order to investigate the structure of the *exactly solvable sector* we make as before the similarity transformation of H by e^{W_0}:

$$\begin{aligned}\tilde{H} &= \mathrm{e}^{-W_0} H\, \mathrm{e}^{W_0} = \frac{1}{2}p^2 - \frac{\partial W_0}{\partial q}\frac{\partial}{\partial q} - 4a\mathcal{M}\sin^2 q \\ &= \frac{1}{2}p^2 - \left(a\sin 2q - \frac{b}{\sin 2q} + g_S\cot q\right)\frac{\partial}{\partial q} - 4a\mathcal{M}\sin^2 q.\end{aligned} \tag{4.15}$$

The eigenfunction of $\tilde{H}$ in the polynomial space $\tilde{\mathcal{V}}_\mathcal{M} = \mathrm{Span}\left[1, \sin^2 q, \ldots, (\sin q)^{2\mathcal{M}}\right]$ can be obtained by substituting an expansion $\tilde{\Psi} = \sum_{k=0}^{\mathcal{M}} \alpha_k (\sin q)^{2k}$, $\alpha_0 = 1$, into the eigenvalue equation $\tilde{H}\tilde{\Psi} = E\tilde{\Psi}$. This again leads to a three-term recursion relation for 'Bender–Dunne' polynomials $\{\alpha_k(E)\}$:

$$(k+1)(2k+1-b+2g_S)\alpha_{k+1} = (2k(1-2a+2g_S) - E)\,\alpha_k + 4a(k-\mathcal{M}-1)\alpha_{k-1}. \tag{4.16}$$

Again $\alpha_k(E)$ is a polynomial in E of degree k and the condition $\alpha_{\mathcal{M}+1} = 0$ gives the characteristic equation for $\tilde{H}$:

$$\alpha_{\mathcal{M}+1} = 0 \Leftrightarrow (2\mathcal{M}(1-2a+2g_S) - E)\,\alpha_\mathcal{M} - 4a\alpha_{\mathcal{M}-1} = 0 \Leftrightarrow \det(\tilde{H} - E) = 0.$$

5. Rational A type Inozemtsev model with one degree of freedom

This is an interesting example which fails to achieve quasi-exact solvability due to the lack of square integrability of the eigenfunction. It is interesting to know how far the algebraic procedures go in parallel with the previous cases. We start with the following pre-superpotential W which is obtained by retaining the single-particle part of W in (2.13):

$$W = W_0 + W_1 \qquad W_0 = \frac{a}{3}q^3 + \frac{b}{2}q^2 \qquad W_1 = \sum_{k=1}^{\mathcal{M}} \log|q - \xi_k|. \tag{5.1}$$

This is cubic in q and leads to a quartic potential of q, see (5.5). The terms containing W_1 in $(\partial W/\partial q)^2 + \partial^2 W/\partial q^2$ are

$$2(aq^2 + bq)\sum_{k=1}^{\mathcal{M}}\frac{1}{q-\xi_k} + 2\sum_{k<l}\frac{1}{q-\xi_k}\frac{1}{q-\xi_l}. \tag{5.2}$$

From the requirement of vanishing residue at $q = \xi_k$, we obtain the rational equations

$$a\xi_k^2 + b\xi_k + \sum_{l\neq k}\frac{1}{\xi_k - \xi_l} = 0 \qquad k, l = 1, \ldots, \mathcal{M} \tag{5.3}$$

and the expression (5.2) is

$$(5.2) = 2a\mathcal{M}q + 2a\sum_{k=1}^{\mathcal{M}}\xi_k + 2b\mathcal{M}.$$

Thus for each real solution $\{\xi_k\}$ of (5.3) we obtain an 'eigenfunction':

$$\mathrm{e}^{W} = \mathrm{e}^{W_0} \prod_{k=1}^{\mathcal{M}} (q - \xi_k) = \mathrm{e}^{(aq^3/3 + bq^2/2)} \prod_{k=1}^{\mathcal{M}} (q - \xi_k) \tag{5.4}$$

of a Hamiltonian:

$$\begin{aligned} H &= \frac{1}{2} p^2 + \frac{1}{2} \left[\left(\frac{\partial W_0}{\partial q} \right)^2 + \frac{\partial^2 W_0}{\partial q^2} \right] + a\mathcal{M}q \\ &= \frac{1}{2} p^2 + \frac{1}{2} q^2 (aq + b)^2 + a \left(\mathcal{M} + \frac{1}{2} \right) q + \frac{b}{2} \end{aligned} \tag{5.5}$$

with energy

$$E = -a \sum_{k=1}^{\mathcal{M}} \xi_k - b\mathcal{M}. \tag{5.6}$$

Let us recall the simple facts about the limiting case of $a = 0$ and $b = -\omega$, $\omega > 0$. Then the Hamiltonian (5.5) becomes that of the simple harmonic oscillator with angular frequency ω and equations (5.3) determine $\{\xi_k\}$ as the zeros of Hermite polynomials [28], with scaling by $\sqrt{\omega}$. This results in the well known eigenfunction with Hermite polynomials (5.4) and the spectrum $E = \omega\mathcal{M}$ (5.6). (The zero-point energy $\omega/2$ is contained in the Hamiltonian (5.5).) Thus, at least for $b < 0$ and $|a/b| \ll 1$, that is, the quartic and the accompanying cubic terms in the potential can be considered as 'perturbations', it is expected that equations (5.3) have real solutions and the above solution generating method would work. However, the 'eigenfunction' (5.4) is not square integrable in the region $(-\infty, +\infty)$ for whichever choice of the sign of $a \neq 0$. One might be tempted to restrict the region to a half line, say $(0, +\infty)$, by introducing a singular potential at the origin, for example, by adding a term $g \log |q|$ to W_0, as in the example in section 3. However, this cannot remedy the situation because of the wrong parity of the aq^3 term.

The $\mathcal{N} = (\mathcal{M} + 1)$th-order differential operator $P_{\mathcal{N}}$ annihilating the 'polynomial space':

$$\mathcal{V}_{\mathcal{M}} = \mathrm{Span}\left[1, q, \ldots, q^k, \ldots, q^{\mathcal{M}}\right] \mathrm{e}^{W_0} \tag{5.7}$$

has a very simple form:

$$P_{\mathcal{N}} = D^{\mathcal{N}} \qquad D = p + \mathrm{i}\frac{\partial W_0}{\partial q}. \tag{5.8}$$

In this case the $\mathcal{N}$-fold supersymmetry generated by $Q = P_{\mathcal{N}}\psi^\dagger$ and $Q^\dagger = P_{\mathcal{N}}^\dagger \psi$ is *spontaneously broken* for $a \neq 0$.

The difference in the algebraic structure from the QES case discussed in section 3 becomes clearer by making the similarity transformation of H by e^{W_0}:

$$\begin{aligned} \tilde{H} &= \mathrm{e}^{-W_0} H \,\mathrm{e}^{W_0} = \frac{1}{2} p^2 - \frac{\partial W_0}{\partial q}\frac{\partial}{\partial q} + a\mathcal{M}q \\ &= \frac{1}{2} p^2 - (aq^2 + bq)\frac{\partial}{\partial q} + a\mathcal{M}q. \end{aligned} \tag{5.9}$$

This maps q^k to q^{k+1}, q^k and q^{k-2}:

$$\tilde{H} q^k = a(\mathcal{M} - k) q^{k+1} - bkq^k - \frac{k(k-1)}{2} q^{k-2}.$$

Since the last term is q^{k-2} instead of q^{k-1}, the three-term recursion relations for the coefficients $\{\alpha_k\}$ in a series solution $\tilde{\Psi} = \sum_{k=0}^{\mathcal{M}} \alpha_k q^k$, $\alpha_0 = 1$ for the eigenvalue equation $\tilde{H}\tilde{\Psi} = E\tilde{\Psi}$ do not hold any more. The $\{\alpha_k(E)\}$ are no longer degree k polynomials in E.

We will not discuss the 'trigonometric A type Inozemtsev model with one degree of freedom', since the single-particle part of (2.14) is simply $W = -\frac{a}{2}\cos 2q$, that is, the 'double sine–Gordon' quantum mechanics. This is a well known example of QES dynamics [27] and is a special case of the model treated in section 4.

6. Rational BC type Inozemtsev model

Following the results of the single-particle case in section 3, we consider the following Hamiltonian:

$$H = \frac{1}{2}\sum_{j=1}^{r} p_j^2 + \frac{1}{2}\sum_{j=1}^{r}\left[\left(\frac{\partial W_0}{\partial q_j}\right)^2 + \frac{\partial^2 W_0}{\partial q_j^2}\right] - 2a\mathcal{M}\sum_{j=1}^{r} q_j^2 \tag{6.1}$$

in which $\mathcal{M}$ is an arbitrary non-negative integer and W_0 is given by (2.11)

$$W_0 = -\sum_{j=1}^{r}\left(\frac{a}{4}q_j^4 + \frac{b}{2}q_j^2\right) + W_{\rm CM} \tag{6.2}$$

$$\begin{aligned} &W_{\rm CM} = g_M\sum_{j<k}^{r}\{\log|q_j - q_k| + \log|q_j + q_k|\} + g_S\sum_{j=1}^{r}\log|q_j| \\ &a > 0 \quad g_S > 0 \quad g_M > 0. \end{aligned} \tag{6.3}$$

The Hamiltonian, as well as W_0, are Coxeter (Weyl) invariants of the BC_r root system. The only difference with the classical supersymmetrizable Inozemtsev model is the added quadratic terms proportional to $\mathcal{M}$. We will consider the model in the principal Weyl chamber

$$q_1 > q_2 > \cdots > q_r > 0. \tag{6.4}$$

A special case of this Hamiltonian with one free parameter other than $\mathcal{M}$ (plus an invisible overall scale factor) was discussed in [29].

In order to show the quasi-exact solvability of the Hamiltonian (6.1) we have to demonstrate that a certain 'exactly solvable sector' is invariant under H. As before, let us first define a 'polynomial space':

$$\mathcal{V}_{\mathcal{M}} = \mathrm{Span}_{0\leqslant n_j\leqslant\mathcal{M},\ 1\leqslant j\leqslant r}\left[(q_1^2)^{n_1}\cdots(q_j^2)^{n_j}\cdots(q_r^2)^{n_r}\right]\mathrm{e}^{W_0}. \tag{6.5}$$

The 'exactly solvable sector' is the *permutation* ($q_j \leftrightarrow q_k$) *invariant subspace* of $\mathcal{V}_{\mathcal{M}}$:

$$\mathcal{V}_{\mathcal{M}}^{G_{BC}} = \{v \in \mathcal{V}_{\mathcal{M}} | gv = v,\ \forall g \in G_{BC}\} \tag{6.6}$$

in which G_{BC} is the Coxeter (Weyl) group of the BC root system. This fact can be seen easily, as in the single-particle case, by similarity transformation:

$$\begin{aligned} \tilde{H} = \mathrm{e}^{-W_0}H\,\mathrm{e}^{W_0} &= \frac{1}{2}\sum_{j=1}^{r}p_j^2 - \sum_{j=1}^{r}\frac{\partial W_0}{\partial q_j}\frac{\partial}{\partial q_j} - 2a\mathcal{M}\sum_{j=1}^{r}q_j^2 \\ &= \frac{1}{2}\sum_{j=1}^{r}p_j^2 - \sum_{j=1}^{r}\frac{\partial W_{\rm CM}}{\partial q_j}\frac{\partial}{\partial q_j} + \sum_{j=1}^{r}(aq_j^3 + bq_j)\frac{\partial}{\partial q_j} - 2a\mathcal{M}\sum_{j=1}^{r}q_j^2 \end{aligned} \tag{6.7}$$

$$\tilde{\mathcal{V}}_{\mathcal{M}} = \mathrm{Span}_{0\leqslant n_j\leqslant\mathcal{M},\ 1\leqslant j\leqslant r}\left[(q_1^2)^{n_1}\cdots(q_j^2)^{n_j}\cdots(q_r^2)^{n_r}\right]. \tag{6.8}$$

The proof of the invariance of $\tilde{\mathcal{V}}_{\mathcal{M}}^{G_{BC}}$ under the added terms $\sum_{j=1}^{r}(aq_j^3+bq_j)\frac{\partial}{\partial q_j} - 2a\mathcal{M}\sum_{j=1}^{r}q_j^2$ is essentially the same as in the single-particle case. As for the CM part, $W_{\rm CM}$, it always decreases the power $\sum_{j=1}^{r}n_j$ by one unit. The Coxeter (Weyl) invariance is necessary and sufficient so that the result remains a polynomial, without developing unwanted poles. Thus, the invariance of $\tilde{\mathcal{V}}_{\mathcal{M}}^{G_{BC}}$ under $\tilde{H}$ and the quasi-exact integrability of H is proved. It is obvious that the elements of $\mathcal{V}_{\mathcal{M}}^{G_{BC}}$ are square integrable since

$$\mathrm{e}^{W_0} = \exp\left(-\sum_{j=1}^{r}(\frac{a}{4}q_j^4 + \frac{b}{2}q_j^2)\right)\prod_{j=1}^{r}(q_j)^{g_S}\prod_{j<k}(q_j^2 - q_k^2)^{g_M} \tag{6.9}$$

and the restriction on the parameters (6.3) and the integration region (6.4) secure finiteness at infinity and at the boundaries of the Weyl chambers.

Thus we have shown that the rational BC type Inozemtsev model can be made QES by adding properly quantized quadratic terms $-2a\mathcal{M}\sum_{j=1}^{r} q_j^2$. The above 'polynomial space' $\mathcal{V}_{\mathcal{M}}$ is annihilated by the following r different $\mathcal{N} = (\mathcal{M}+1)$th-order commuting differential operators $P_{\mathcal{N}}^{(j)}$:

$$P_{\mathcal{N}}^{(j)} = \prod_{k=0}^{\mathcal{N}-1}\left(D_j + \mathrm{i}\frac{k}{q_j}\right) \qquad D_j = p_j + \mathrm{i}\frac{\partial W_0}{\partial q_j}, \tag{6.10}$$

$$P_{\mathcal{N}}^{(j)}\mathcal{V}_{\mathcal{M}} = 0 \qquad [P_{\mathcal{N}}^{(j)}, P_{\mathcal{N}}^{(k)}] = 0 \qquad j,k = 1,\ldots,r. \tag{6.11}$$

The $\mathcal{N}$-fold supersymmetry is generated by

$$Q = \sum_{j=1}^{r} P_{\mathcal{N}}^{(j)}\psi_j^\dagger \qquad Q^\dagger = \sum_{j=1}^{r}(P_{\mathcal{N}}^{(j)})^\dagger\psi_j \tag{6.12}$$

in which ψ_j and $\psi_j^\dagger$ are the annihilation and creation operators of the jth fermion.

7. Trigonometric BC type Inozemtsev model

Following the results of the single-particle case in section 4, we consider the following Hamiltonian:

$$H = \frac{1}{2}\sum_{j=1}^{r} p_j^2 + \frac{1}{2}\sum_{j=1}^{r}\left[\left(\frac{\partial W_0}{\partial q_j}\right)^2 + \frac{\partial^2 W_0}{\partial q_j^2}\right] - 4a\mathcal{M}\sum_{j=1}^{r}\sin^2 q_j \tag{7.1}$$

in which $\mathcal{M}$ is an arbitrary non-negative integer and W_0 is given by (2.12)

$$W_0 = \sum_{j=1}^{r}\left(-\frac{a}{2}\cos 2q_j + \frac{b}{2}\log|\cot q_j|\right) + W_{\mathrm{CM}} \tag{7.2}$$

$$W_{\mathrm{CM}} = g_M\sum_{j<k}^{r}\left\{\log|\sin(q_j - q_k)| + \log|\sin(q_j + q_k)|\right\} + g_S\sum_{j=1}^{r}\log|\sin q_j|. \tag{7.3}$$

All the parameters a, b, g_M and g_S are real and they satisfy

$$g_S > 0 \qquad g_M > 0 \qquad g_S > \frac{b}{2} > 0. \tag{7.4}$$

Let us recall here the argument in section 4 for dropping a $g_L\log|\sin 2q_j|$ term in favour of a $b\log|\cot q_j|$ term. The Hamiltonian, as well as W_0, are Coxeter (Weyl) invariants of the BC_r root system. The only difference with the classical supersymmetrizable Inozemtsev model is the added $\sin^2 q_j$ terms proportional to $\mathcal{M}$. We will consider the quantum mechanical model in the principal Weyl alcove

$$\pi/2 > q_1 > q_2 > \cdots > q_r > 0 \tag{7.5}$$

due to the periodicity and Coxeter (Weyl) invariance of the model. As in the single-particle case, we will demonstrate that a certain 'exactly solvable sector' is invariant under H. As before, let us first define a 'polynomial space'

$$\mathcal{V}_{\mathcal{M}} = \mathrm{Span}_{0\leqslant n_j\leqslant\mathcal{M},\ 1\leqslant j\leqslant r}\left[(\sin^2 q_1)^{n_1}\cdots(\sin^2 q_j)^{n_j}\cdots(\sin^2 q_r)^{n_r}\right]\mathrm{e}^{W_0} \tag{7.6}$$

or equivalently

$$\mathcal{V}_{\mathcal{M}} = \mathrm{Span}_{-\mathcal{M}\leqslant n'_j\leqslant\mathcal{M},\ 1\leqslant j\leqslant r}\left[\cos 2\left(\sum_{j=1}^{r} n'_j q_j\right)\right]\mathrm{e}^{W_0}. \tag{7.7}$$

The 'exactly solvable sector' is the *permutation* ($q_j \leftrightarrow q_k$) *invariant subspace* of $\mathcal{V}_{\mathcal{M}}$:

$$\mathcal{V}_{\mathcal{M}}^{G_{BC}} = \{v \in \mathcal{V}_{\mathcal{M}} | gv = v, \forall g \in G_{BC}\} \tag{7.8}$$

in which G_{BC} is the Coxeter (Weyl) group of the BC root system. All these functions are square integrable, since the integration region is finite and the possible singularities in e^{W_0} at the boundary (7.5):

$$\mathrm{e}^{W_0} = \exp\left(-\frac{a}{2}\sum_{j=1}^{r}\cos 2q_j\right)\prod_{j=1}^{r}(\sin q_j)^{g_S}(\cot q_j)^{b/2}\prod_{k<l}^{r}[\sin(q_k - q_l)\sin(q_k + q_l)]^{g_M} \tag{7.9}$$

are taken care of by the restriction on the parameters (7.4). The quasi-exact solvability can be shown by the similarity transformation

$$\begin{aligned}\tilde{H} = \mathrm{e}^{-W_0} H\,\mathrm{e}^{W_0} &= \frac{1}{2}\sum_{j=1}^{r} p_j^2 - \sum_{j=1}^{r}\frac{\partial W_0}{\partial q_j}\frac{\partial}{\partial q_j} - 4a\mathcal{M}\sum_{j=1}^{r}\sin^2 q_j \\ &= \frac{1}{2}\sum_{j=1}^{r} p_j^2 - \sum_{j=1}^{r}\frac{\partial W_{\mathrm{CM}}}{\partial q_j}\frac{\partial}{\partial q_j} - \sum_{j=1}^{r}\left(a\sin 2q_j - \frac{b}{\sin 2q_j}\right)\frac{\partial}{\partial q_j} \\ &\quad -4a\mathcal{M}\sum_{j=1}^{r}\sin^2 q_j\end{aligned} \tag{7.10}$$

$$\tilde{\mathcal{V}}_{\mathcal{M}} = \mathrm{Span}_{0\leqslant n_j\leqslant\mathcal{M},\ 1\leqslant j\leqslant r}\left[(\sin^2 q_1)^{n_1}\cdots(\sin^2 q_j)^{n_j}\cdots(\sin^2 q_r)^{n_r}\right] \tag{7.11}$$

$$= \mathrm{Span}_{-\mathcal{M}\leqslant n'_j\leqslant\mathcal{M},\ 1\leqslant j\leqslant r}\left[\cos 2\left(\sum_{j=1}^{r} n'_j q_j\right)\right]. \tag{7.12}$$

The corresponding 'exactly solvable sector' is the *permutation* ($q_j \leftrightarrow q_k$) *invariant subspace* of $\tilde{\mathcal{V}}_{\mathcal{M}}$:

$$\tilde{\mathcal{V}}_{\mathcal{M}}^{G_{BC}} = \left\{v \in \tilde{\mathcal{V}}_{\mathcal{M}} | gv = v, \forall g \in G_{BC}\right\}. \tag{7.13}$$

As for the CM part, the above Hamiltonian is lower triangular [16] in the basis (B.6) of $\tilde{\mathcal{V}}_{\mathcal{M}}^{G_{BC}}$. The lower triangularity is stronger than the invariance of the polynomial space under the Hamiltonian. An outline of the proof is given in appendix B. As for the added part

$$-\sum_{j=1}^{r}\left(a\sin 2q_j - \frac{b}{\sin 2q_j}\right)\frac{\partial}{\partial q_j} - 4a\mathcal{M}\sum_{j=1}^{r}\sin^2 q_j$$

the proof of the invariance of the polynomial space (7.11) is essentially the same as in the single-particle case. Thus the quasi-exact solvability of the trigonometric BC type Inozemtsev model is established.

The above 'polynomial space' $\mathcal{V}_{\mathcal{M}}$ (7.6) is annihilated by the following r different $\mathcal{N} = (\mathcal{M}+1)$th-order commuting differential operators $P_{\mathcal{N}}^{(j)}$:

$$P_{\mathcal{N}}^{(j)} = \prod_{k=0}^{\mathcal{N}-1}(D_j + 2\mathrm{i}k\cot 2q_j) \qquad D_j = p_j + \mathrm{i}\frac{\partial W_0}{\partial q_j} \tag{7.14}$$

$$P_{\mathcal{N}}^{(j)}\mathcal{V}_{\mathcal{M}} = 0 \qquad [P_{\mathcal{N}}^{(j)}, P_{\mathcal{N}}^{(k)}] = 0 \qquad j,k = 1,\ldots,r. \tag{7.15}$$

The $\mathcal{N}$-fold supersymmetry is generated by

$$Q = \sum_{j=1}^{r} P_{\mathcal{N}}^{(j)}\psi_j^\dagger \qquad Q^\dagger = \sum_{j=1}^{r}(P_{\mathcal{N}}^{(j)})^\dagger\psi_j$$

in which ψ_j and $\psi_j^\dagger$ are the annihilation and creation operators of the jth fermion.

8. Trigonometric A type Inozemtsev model

This has a much simpler Hamiltonian than the previous one:

$$H = \frac{1}{2}\sum_{j=1}^{r} p_j^2 + \frac{1}{2}\sum_{j=1}^{r}\left[\left(\frac{\partial W_0}{\partial q_j}\right)^2 + \frac{\partial^2 W_0}{\partial q_j^2}\right] - 4a\mathcal{M}\sum_{j=1}^{r}\sin^2 q_j \tag{8.1}$$

in which $\mathcal{M}$ is an arbitrary non-negative integer and W_0 is given by (2.14)

$$W_0 = -\frac{a}{2}\sum_{j=1}^{r}\cos 2q_j + W_{\mathrm{CM}} \qquad W_{\mathrm{CM}} = g\sum_{j<k}^{r}\log|\sin(q_j - q_k)|. \tag{8.2}$$

There are only two real coupling constants a and g and we require

$$g > 0 \tag{8.3}$$

for square integrability of the eigenfunctions of the form (8.5). The Hamiltonian and W_0 are invariant under any transpositions $(q_j, p_j) \leftrightarrow (q_k, p_k)$, which form the Coxeter group of A_{r-1}. Thus we consider the quantum mechanics in the fundamental Weyl alcove

$$\pi > q_1 > q_2 > \cdots > q_r > 0. \tag{8.4}$$

Reflecting the simple form of the Hamiltonian, the 'exactly solvable sector' which is left invariant under H has a simpler structure than those in the previous cases. Let us first define a space of truncated Fourier series with two units:

$$\mathcal{V}_{\mathcal{M}} = \mathrm{Span}_{-\mathcal{M}\leqslant n_j\leqslant\mathcal{M},\ 1\leqslant j\leqslant r}\left[\mathrm{e}^{2\mathrm{i}\sum_{j=1}^{r} n_j q_j}\right]\mathrm{e}^{W_0}. \tag{8.5}$$

The 'exactly solvable sector' is the *permutation* $(q_j \leftrightarrow q_k)$ *invariant subspace* of $\mathcal{V}_{\mathcal{M}}$:

$$\mathcal{V}_{\mathcal{M}}^{G_A} = \left\{v \in \mathcal{V}_{\mathcal{M}} | gv = v,\ \forall g \in G_A\right\} \tag{8.6}$$

in which G_A is the Coxeter (Weyl) group of the A type root system. The quasi-exact solvability of the Hamiltonian (8.1) is again easily verified by the similarity transformation:

$$\begin{aligned}\tilde{H} = \mathrm{e}^{-W_0}H\,\mathrm{e}^{W_0} &= \frac{1}{2}\sum_{j=1}^{r} p_j^2 - \sum_{j=1}^{r}\frac{\partial W_0}{\partial q_j}\frac{\partial}{\partial q_j} - 4a\mathcal{M}\sum_{j=1}^{r}\sin^2 q_j \\ &= \frac{1}{2}\sum_{j=1}^{r} p_j^2 - \sum_{j=1}^{r}\frac{\partial W_{\mathrm{CM}}}{\partial q_j}\frac{\partial}{\partial q_j} - a\sum_{j=1}^{r}\sin 2q_j\frac{\partial}{\partial q_j} - 4a\mathcal{M}\sum_{j=1}^{r}\sin^2 q_j\end{aligned} \tag{8.7}$$

$$\tilde{\mathcal{V}}_{\mathcal{M}} = \mathrm{Span}_{-\mathcal{M}\leqslant n_j\leqslant\mathcal{M},\ 1\leqslant j\leqslant r}\left[\mathrm{e}^{2\mathrm{i}\sum_{j=1}^{r} n_j q_j}\right]. \tag{8.8}$$

The lower triangularity of the CM part of the Hamiltonian in the basis (B.1) of (8.8) was proven originally by Sutherland [7]. For the additional part

$$-a\sum_{j=1}^{r}\sin 2q_j\frac{\partial}{\partial q_j}-4a\mathcal{M}\sum_{j=1}^{r}\sin^2 q_j$$

the proof that it leaves the space of the truncated Fourier series (8.8) invariant is rather elementary.

The above 'polynomial space' $\mathcal{V}_{\mathcal{M}}$ (8.5) is annihilated by the following r different $\mathcal{N}=(2\mathcal{M}+1)$th-order commuting differential operators $P_{\mathcal{N}}^{(j)}$:

$$P_{\mathcal{N}}^{(j)}=\prod_{k=-\mathcal{M}}^{\mathcal{M}}(D_j+2\mathrm{i}k) \qquad D_j=p_j+\mathrm{i}\frac{\partial W_0}{\partial q_j} \tag{8.9}$$

$$P_{\mathcal{N}}^{(j)}\mathcal{V}_{\mathcal{M}}=0 \qquad [P_{\mathcal{N}}^{(j)},P_{\mathcal{N}}^{(k)}]=0 \qquad j,k=1,\ldots,r. \tag{8.10}$$

The $\mathcal{N}$-fold supersymmetry is generated by

$$Q=\sum_{j=1}^{r}P_{\mathcal{N}}^{(j)}\psi_j^{\dagger} \qquad Q^{\dagger}=\sum_{j=1}^{r}(P_{\mathcal{N}}^{(j)})^{\dagger}\psi_j$$

in which ψ_j and $\psi_j^{\dagger}$ are the annihilation and creation operators of the jth fermion.

We will not discuss the quasi-exact solvability of the multi-particle rational A type Inozemtsev model, since its wavefunctions are not square integrable as seen in section 5. One could say that the $\mathcal{N}$-fold supersymmetry is spontaneously broken in this case.

Let us summarize that the quasi-exact solvability of quantum Inozemtsev models discussed in sections 6, 7 and 8 is a consequence of the exact solvability of the quantum CM models and the quasi-exact solvability of the added single-particle-like interactions.

9. Comments and discussion

It should be stressed that the present method for showing quasi-exact solvability of single-particle systems developed in sections 3–5 does not depend on any existing methods or criteria for QES, see for example, [30]. As shown in section 5 it also gives the known exact solutions when the QES system tends to the harmonic oscillator.

An interesting question along the line of arguments in this paper is the 'hierarchy problem' of QES systems, as in the completely integrable systems. For example, the Inozemtsev models have higher conserved quantities $\mathrm{Tr}(L^k)$ obtained from the Lax pairs in appendix A. They define new classical and quantum Hamiltonian systems. Can the quantum version of the higher members of the hierarchy be deformed to be QES?

So far, elliptic CM models have defied various attempts to construct quantum theory based on a Hilbert space, although existence of mutually commuting operators are known for the A type models [31]. In analogy with the present arguments, it is quite natural to expect the quantum elliptic CM models to be QES [32, 33] rather than exactly integrable.

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Appendix A. Lax pairs for classical Inozemtsev models

Here we present the Lax pairs for classical Inozemtsev models in the same notation as is used in the main text as evidence for their classical integrability. As mentioned in the introduction only a certain subset of classical Inozemtsev models can be made QES. We focus on the supersymmetrizable Inozemtsev models for simplicity of presentation. For the full content of Lax pairs of classical Inozemtsev models we refer to the original paper by Inozemtsev and Meshchryakov [5] and for the universal Lax pairs for CM models in general see [11, 18].

The Lax pair consists of a pair of $2r \times 2r$ (r rank) matrices L and M such that the canonical equations of motion can be expressed in a matrix form

$$\dot{L} = [L, M]$$

and a sufficient number of classical conserved quantities can be obtained as the trace of powers of L, $\mathrm{Tr}(L^k)$. The Hamiltonian is $H \propto \mathrm{Tr}(L^2)$.

A.1. BC type models

The following Lax pair applies for the models presented in section 2.5 for the rational as well as trigonometric models for a proper choice of functions, x, ν, etc, as listed below. The pair of matrices decomposes into diagonal and off-diagonal matrices:

$$L = P + X \qquad M = D + Y. \tag{A.1}$$

The diagonal matrices P and D are of the form

$$P = \sum_{j=1}^{r} p_j (E_{j,j} - E_{j+r,j+r}) \qquad D = \sum_{j=1}^{r} D_j (E_{j,j} + E_{j+r,j+r}) \tag{A.2}$$

in which $E_{j,k}$ is the usual matrix unit $(E_{j,k})_{lm} = \delta_{lj}\delta_{mk}$. The diagonal-free matrices X and Y have the form

$$\begin{aligned} X = {} & \mathrm{i}g_M \sum_{j\neq k} x(q_j - q_k) E_{j,k} + \mathrm{i}g_M \sum_{j\neq k} x(q_j + q_k) E_{j,k+r} \\ & + \mathrm{i}g_M \sum_{j\neq k} x(-q_j - q_k) E_{j+r,k} + \mathrm{i}g_M \sum_{j\neq k} x(-q_j + q_k) E_{j+r,k+r} \\ & + 2\mathrm{i} \sum_{j} \nu(q_j) E_{j,j+r} - 2\mathrm{i} \sum_{j} \nu(q_j) E_{j+r,j} \end{aligned} \tag{A.3}$$

$$\begin{aligned} Y = {} & \mathrm{i}g_M \sum_{j\neq k} y(q_j - q_k) E_{j,k} + \mathrm{i}g_M \sum_{j\neq k} y(q_j + q_k) E_{j,k+r} \\ & + \mathrm{i}g_M \sum_{j\neq k} y(-q_j - q_k) E_{j+r,k} + \mathrm{i}g_M \sum_{j\neq k} y(-q_j + q_k) E_{j+r,k+r} \\ & + \mathrm{i} \sum_{j} \nu'(q_j) E_{j,j+r} + \mathrm{i} \sum_{j} \nu'(q_j) E_{j+r,j}. \end{aligned} \tag{A.4}$$

The diagonal elements of D are given by

$$D_j = -\mathrm{i}g_M \sum_{k\neq j}^{r} \big(z(q_j - q_k) + z(q_j + q_k) \big) - \mathrm{i} \sum_{j=1}^{r} \tau(q_j). \tag{A.5}$$

Some functions are related to each other:

$$y(u) = \mathrm{d}x\,(u)/\mathrm{d}u \qquad z(u) = x(u)^2 + \mathrm{const} \qquad \tau(u) = 2x(2u)\nu(u) + \mathrm{const}. \tag{A.6}$$

The rational and trigonometric models correspond to the following choice of functions:

(1) Rational model:

$$x(u) = \frac{1}{u} \qquad z(u) = \frac{1}{u^2} \qquad v(u) = -(au^3 + bu) + \frac{g_S}{u} \tag{A.7}$$

where a, b and g_S are real coupling constants.

(2) Trigonometric model:

$$x(u) = \cot u \qquad z(u) = \frac{1}{\sin^2 u} \qquad v(u) = a\sin 2u - \frac{b}{\sin 2u} + g_S \cot u \tag{A.8}$$

where a, b and g_S are real coupling constants.

The functions x and v correspond to those appearing in $\partial W/\partial q$.

A.2. A type models

The Lax pair is again a pair of $2r \times 2r$ ($r-1$ is the rank) matrices, with the decomposition

$$L = P + X \qquad M = D + Y$$

in which P and D are diagonal matrices of the form

$$P = \sum_{j=1}^{r} p_j (E_{j,j} - E_{j+r,j+r}) \qquad D = \sum_{j=1}^{r} D_j (E_{j,j} + E_{j+r,j+r})$$

and X and Y are diagonal-free matrices of the form

$$\begin{aligned} X = \mathrm{i}g \sum_{j\neq k} x(q_j - q_k)E_{j,k} + \mathrm{i}g \sum_{j\neq k} x(-q_j + q_k)E_{j+r,k+r} \\ +2\sum_j \kappa(q_j)E_{j,j+r} + 2\sum_j \kappa(-q_j)E_{j+r,j} \end{aligned} \tag{A.9}$$

$$\begin{aligned} Y = \mathrm{i}g \sum_{j\neq k} y(q_j - q_k)E_{j,k} + \mathrm{i}g \sum_{j\neq k} y(-q_j + q_k)E_{j+r,k+r} \\ +\sum_j \kappa'(q_j)E_{j,j+r} + \sum_j \kappa'(-q_j)E_{j+r,j}. \end{aligned} \tag{A.10}$$

The diagonal elements of D are given by

$$D_j = -\mathrm{i}g \sum_{k\neq j}^{r} z(q_j - q_k). \tag{A.11}$$

The rational and trigonometric models correspond to the following choice of functions:

(1) Rational model:

$$x(u) = \frac{1}{u} \qquad z(u) = \frac{1}{u^2} \qquad \kappa(u) = au^2 + bu \tag{A.12}$$

where a and b are arbitrary real coupling constants.

(2) Trigonometric model:

$$x(u) = \frac{1}{\sin u} \qquad z(u) = \frac{1}{\sin^2 u} \qquad \kappa(u) = a\sin 2u \tag{A.13}$$

where a is the arbitrary real coupling constant.

Appendix B. Lower triangularity

Here we show the details of the argument that the CM part of the similarity transformed Hamiltonian (7.10) is lower triangular in the basis (7.13). The triangularity of the trigonometric CM model is proved universally in [16] by using the Coxeter (Weyl) invariant basis:

$$\phi_\lambda(q) \equiv \sum_{\mu \in W(\lambda)} \mathrm{e}^{2\mathrm{i}\mu\cdot q} \tag{B.1}$$

in which λ is a dominant weight:

$$\lambda = \sum_{j=1}^{r} m_j \lambda_j \qquad m_j \in \boldsymbol{Z}_+ \quad \lambda_j : \text{fundamental weight} \tag{B.2}$$

and $W(\lambda)$ is the orbit of λ by the action of the Weyl group:

$$W(\lambda) = \{\mu \in \Lambda(\Delta) | \mu = g(\lambda),\ \forall g \in G_\Delta\} \qquad \Lambda(\Delta) : \text{weight lattice of } \Delta. \tag{B.3}$$

The above $\phi_\lambda(q)$ is Coxeter invariant. The set of functions $\{\phi_\lambda\}$ has an order $>$:

$$|\lambda|^2 > |\lambda'|^2 \Rightarrow \phi_\lambda > \phi_{\lambda'}. \tag{B.4}$$

For the BC_r root system, the set of weights $W(\lambda)$ is symmetric:

$$\mu \in W(\lambda) \Leftrightarrow -\mu \in W(\lambda). \tag{B.5}$$

Thus the Coxeter invariant basis (B.1) for BC type root system can be rewritten as

$$\phi'_\lambda(q) \equiv \sum_{\mu \in W(\lambda)} \cos(2\mu \cdot q). \tag{B.6}$$

All the fundamental weights, except for the spinor weights of B_r, are integral. That is, so long as the dominant weight λ above contains the fundamental spinor weights in *even* multiples, all the $\mu \cdot q$ in (B.1) have the form

$$\mu \cdot q = \sum_{j=1}^{r} k_j q_j \qquad k_j \in \boldsymbol{Z}_+.$$

Therefore the basis (B.6) has the form (7.13) used in section 7. Moreover, after the application of $\tilde{H}$ on the basis functions (7.13), those appearing as lower terms also have the same property of having *even* numbers of fundamental spinor weights and can be expressed in the same form (7.13). This completes the proof of the lower triangularity of the Hamiltonian (7.10) in the space (7.13).

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